Problem 2.16

A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2} \quad (a \le r \le b)$$

(Fig. 2.25). Find the electric field in the following three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$. Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

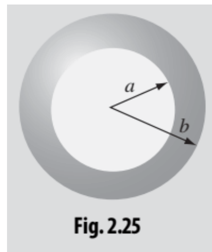


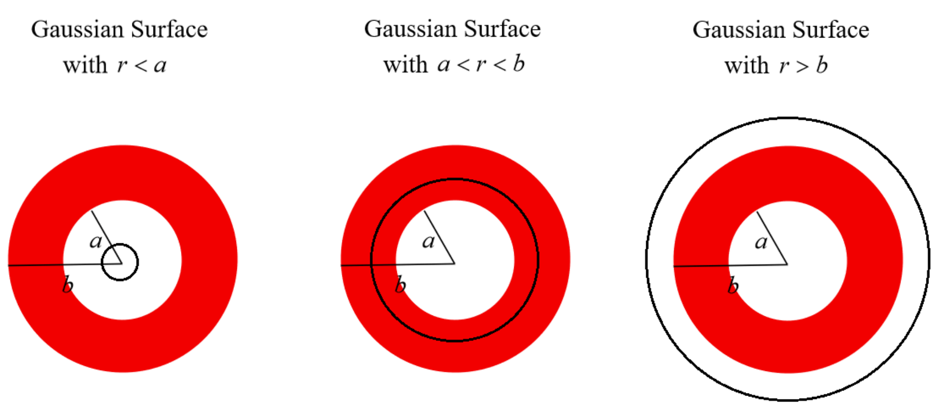
Fig. 2.25

Solution

One of the governing equations for the electric field in vacuum is Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Normally the curl of $\mathbf{E}$ is also necessary to determine $\mathbf{E}$, but because of the spherical symmetry, the divergence is sufficient. Both sides will be integrated over the volume of a (black) concentric spherical Gaussian surface with radius r. Three cases need to be considered: (1) $r < a$ and (2) $a < r < b$ and (3) $r > b$.



Integrate both sides of Gauss's law over the volume of the spherical Gaussian surface.

$$\iiint_{x_0^2+y_0^2+z_0^2\le r^2} \nabla\cdot\mathbf{E}\,dV_0 = \iiint_{x_0^2+y_0^2+z_0^2\le r^2} \frac{\rho}{\epsilon_0}\,dV_0$$

$$= \frac{1}{\epsilon_0}\iiint_{x_0^2+y_0^2+z_0^2\le r^2} \rho\,dV_0$$

$$= \begin{cases} \frac{1}{\epsilon_0}\int_0^\pi\int_0^{2\pi}\int_0^r (0)(r_0^2\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0) & \text{if } r<a \\ \frac{1}{\epsilon_0}\int_0^\pi\int_0^{2\pi}\int_a^r \frac{k}{r_0^2}(r_0^2\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0) & \text{if } a<r<b \\ \frac{1}{\epsilon_0}\int_0^\pi\int_0^{2\pi}\int_a^b \frac{k}{r_0^2}(r_0^2\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0) & \text{if } r>b \end{cases}$$

Apply the divergence theorem on the left side and evaluate the integrals on the right side.

$$\oiint_{x_0^2+y_0^2+z_0^2=r^2} \mathbf{E}\cdot d\mathbf{S}_0 = \begin{cases} 0 & \text{if } r<a \\ \frac{k}{\epsilon_0}\left(\int_0^\pi \sin\theta_0\,d\theta_0\right)\left(\int_0^{2\pi} d\phi_0\right)\left(\int_a^r dr_0\right) & \text{if } a<r<b \\ \frac{k}{\epsilon_0}\left(\int_0^\pi \sin\theta_0\,d\theta_0\right)\left(\int_0^{2\pi} d\phi_0\right)\left(\int_a^b dr_0\right) & \text{if } r>b \end{cases}$$

Because of the spherical symmetry, the electric field is expected to be entirely radial: $\mathbf{E} = E(r)\hat{\mathbf{r}}$. Note also that the direction of $d\mathbf{S}$ is the outward unit vector to the Gaussian surface.

$$\oiint_{r_0^2=r^2} [E(r_0)\hat{\mathbf{r}}_0]\cdot(\hat{\mathbf{r}}_0\,dS_0) = \begin{cases} 0 & \text{if } r<a \\ \frac{k}{\epsilon_0}(2)(2\pi)(r-a) & \text{if } a<r<b \\ \frac{k}{\epsilon_0}(2)(2\pi)(b-a) & \text{if } r>b \end{cases}$$

Evaluate the dot product.

$$\oiint_{r_0=r} E(r)\,dS_0 = \begin{cases} 0 & \text{if } r<a \\ \frac{4\pi k}{\epsilon_0}(r-a) & \text{if } a<r<b \\ \frac{4\pi k}{\epsilon_0}(b-a) & \text{if } r>b \end{cases}$$

$E(r)$ is constant on the spherical Gaussian surface, so it can be pulled in front of the integral.

$$E(r) \oiint_{r_0=r} dS_0 = \begin{cases} 0 & \text{if } r < a \\ \frac{4\pi k}{\epsilon_0}(r-a) & \text{if } a < r < b \\ \frac{4\pi k}{\epsilon_0}(b-a) & \text{if } r > b \end{cases}$$

Evaluate the surface integral.

$$E(r)(4\pi r^2) = \begin{cases} 0 & \text{if } r < a \\ \frac{4\pi k}{\epsilon_0}(r-a) & \text{if } a < r < b \\ \frac{4\pi k}{\epsilon_0}(b-a) & \text{if } r > b \end{cases}$$

Solve for $E(r)$.

$$E(r) = \begin{cases} 0 & \text{if } r < a \\ \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right) & \text{if } a < r < b \\ \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right) & \text{if } r > b \end{cases}$$

Therefore, the electric field around a thick spherical shell with charge density $\rho = k/r^2$ for $a \leq r \leq b$ is

$$\mathbf{E} = \begin{cases} \mathbf{0} & \text{if } r < a \\ \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{\mathbf{r}} & \text{if } a < r < b \\ \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right)\hat{\mathbf{r}} & \text{if } r > b \end{cases}.$$

Below is a plot of $|\mathbf{E}| = E(r)$ versus r for the special case that $b = 2a$.